

RINGS WHOSE MODULES ARE WEAKLY SUPPLEMENTED ARE PERFECT.

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ABSTRACT. In this note we show that a ring R is left perfect if and only if every left R -module is weakly supplemented if and only if R is semilocal and the radical of the countably infinite free left R -module has a weak supplement.

H. Bass characterized in [1] those ring R whose left R -modules have projective covers and termed them *left perfect rings*. He characterized them as those semilocal rings which have a left t -nilpotent Jacobson radical $\text{Jac}(R)$. Bass' *semiperfect rings* are those whose finitely generated left (or right) R -modules have projective covers and can be characterized as those semilocal rings which have the property that idempotents lift modulo $\text{Jac}(R)$. Kasch and Mares transferred in [3] the notions of perfect and semiperfect rings to modules and characterized semiperfect modules by a lattice theoretical condition as follows: a module M is called *supplemented* if for any submodule N of M there exists a submodule L of M minimal with respect to $M = N + L$. The left perfect rings are then shown to be exactly those rings whose left R -modules are supplemented while the semiperfect rings are those whose finitely generated left R -modules are supplemented. Equivalently it is enough for a ring R to be semiperfect if the left (or right) R -module R is supplemented. Recall that a submodule N of a module M is called *small*, denoted by $N \ll M$, if $N + L \neq M$ for all proper submodules L of M . Weakening the “supplemented”-condition one calls a module *weakly supplemented* if for every submodule N of M there exists a submodule L of M with $N + L = M$ and $N \cap L \ll M$. The semilocal rings R are precisely those rings whose finitely generated left (or right) R -modules are weakly supplemented. Again it is enough that R is weakly supplemented as left (or right) R -module. Semilocal rings which are not semiperfect are examples of weakly supplemented modules which are not supplemented. In this note we prove that if R is semilocal and the radical of the countably infinite free left R -module has a weak supplement, then R has to be left perfect, i.e. every left R -module is supplemented.

Throughout this note all rings are associative with unit and modules are considered to be unital. An ideal I of a ring R is called left t -nilpotent if for any family $\{a_i\}_{i \in \mathbb{N}}$ of elements of R there exists $n > 0$ such that $a_1 a_2 \cdots a_n = 0$. A ring R is left perfect if and only if it

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is semilocal and $\text{Jac}(R)$ is left t -nilpotent. Recall that an infinite family $\{A_\lambda \mid \lambda \in \Lambda\}$ of left ideals of R is called *left vanishing* if given any sequence a_1, a_2, \dots , with $a_i \in A_{\lambda_i}$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$, there exists a number $n \geq 1$ for which $a_1 a_2 a_3 \cdots a_n = 0$. Ware and Zelmanowitz proved in [5, Theorem 1] that for any endomorphism $f \in \text{End}(F)$ of a free module F and endomorphism which belongs to the Jacobson radical $\text{Jac}(\text{End}(F))$, the family $\{\pi_\lambda(\text{Im}(f))\}_\Lambda$ of left ideals of R is left vanishing. Using this result we can prove our main theorem:

Theorem 1. *The following statements are equivalent for a ring R :*

- (a) *Every left R -module is weakly supplemented;*
- (b) *$R^{(\mathbb{N})}$ is weakly supplemented;*
- (c) *R is semilocal and $\text{Rad}(R^{(\mathbb{N})})$ has a weak supplement in $R^{(\mathbb{N})}$.*
- (d) *R is left perfect.*

Proof. (d) \Rightarrow (a) \Rightarrow (b) \Rightarrow (c) is clear and we just need to show (c) \Rightarrow (d): Set $F = R^{(\mathbb{N})}$ and denote $J = \text{Jac}(R)$. Suppose that R is semilocal and $JF = \text{Rad}(F)$ has a weak supplement in F . Let L be a weak supplement of JF in F , i.e. $JF + L = F$ and $JF \cap L \ll F$. Then $R = \pi_i(JF + L) = J + \pi_i(L) = \pi_i(L)$ for any $i \in \mathbb{N}$ implies that there exists $x_i \in L$ such that $\pi_i(x_i) = 1$. Let $\{a_i\}_{i \in \mathbb{N}}$ be any family of elements of J then $a_i x_i \in JL \subseteq JF \cap L \ll F$ and $\pi_i(a_i x_i) = a_i$ for any $i \in \mathbb{N}$. Define $f : F \rightarrow F$ by $f(z) = \sum_{i \in \mathbb{N}} z_i a_i x_i$ for all $z \in F$. Since $\text{Im}(f) \ll F$, we get by Ware and Zelmanowitz's Theorem [5, Theorem 1] that $\{\pi_i(JL)\}_{i \in \mathbb{N}}$ is left vanishing. Thus there exists $n > 0$ such that

$$a_1 a_2 \cdots a_n = \pi_1(a_1 x_1) \pi_2(a_2 x_2) \cdots \pi_n(a_n x_n) = 0.$$

This shows that $\text{Jac}(R)$ is left t -nilpotent and hence R is left perfect. \square

Let $\sigma[M]$ denote the Wisbauer category of a module M , i.e. the full category of $R\text{-Mod}$ consisting of submodules of quotients of direct sums of copies of M . A module M is called a self-generator if any of its submodules is an image of a direct sum of copies of M .

Corollary 2. *Let M be a finitely generated, self-projective, self-generator. Then every module in $\sigma[M]$ is weakly supplemented if and only if $\text{End}(M)$ is left perfect.*

Proof. By [6, 18.3] M is projective in $\sigma[M]$ and by [6, 8.5] M is a generator in $\sigma[M]$. Hence by [6, 46.2] the functor $\text{Hom}(M, -)$ is a Morita equivalence between $\sigma[M]$ and $\text{End}(M)\text{-Mod}$. Thus every module in $\sigma[M]$ is weakly supplemented if and only if every left $\text{End}(M)$ -module is weakly supplemented, which holds if and only if $\text{End}(M)$ is left perfect by the Theorem. \square

We finish the paper with a comment on weak supplements of images of endomorphisms. Recall that a left R -module M is called semi-projective if for any endomorphism $f \in S = \text{End}(M)$ we have $Sf = \text{Hom}(M, \text{Im}(f))$. The module M is called π -projective if for any submodules N, L of M with $M = N + L$ we have $S = \text{Hom}(M, N) + \text{Hom}(M, L)$.

Proposition 3. *Suppose M is a semi-projective and π -projective R -module. Then $S/\text{Jac}(S)$ is regular if and only if $\text{Im}(f)$ has a weak supplement in M for each $f \in S$.*

Proof. (\Rightarrow) Let $f \in S$. By hypothesis there is a $g \in S$ such that $f - fgf \in J(S)$. We have $\text{Im}(f) + \text{Im}(1 - fg) = M$. It is easy to see that $\text{Im}(f) \cap \text{Im}(1 - fg) \subseteq \text{Im}(f - fgf)$, but since $f - fgf \in \text{Jac}(S)$ we have $\text{Im}(f - fgf) \ll M$. Hence $\text{Im}(1 - fg)$ is a weak supplement of $\text{Im}(f)$ in M .

(\Leftarrow) Let $f \in S$ and K be a weak supplement of $\text{Im}(f)$ in M . Since M is semi-projective and π -projective we have $S = \text{Hom}(M, \text{Im}(f)) + \text{Hom}(M, K) = Sf + \text{Hom}(M, K)$. Since $Sf \cap \text{Hom}(M, K) = \text{Hom}(M, \text{Im}(f) \cap K)$ and $\text{Im}(f) \cap K \ll M$, we get $Sf \cap \text{Hom}(M, K) \subseteq \text{Jac}(S)$. Thus Sf has weak supplement for all f , which implies $S/\text{Jac}(S)$ being von Neumann regular by [4, 3.18]. \square

The last proposition generalizes [4, 3.18]. Also as a consequence we conclude that the endomorphism ring of a semi-projective, π -projective weakly supplemented module is regular modulo its Jacobson radical.

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